Segmentation of Polycrystalline Images Using Voronoi Diagrams

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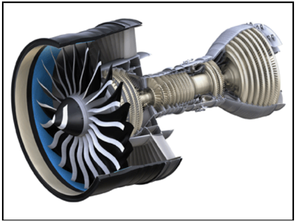
Introduction & Motivation

- Material development is essential to solving problems our world faces.
  - Superalloys: Industrial engineering applications such as aerospace and marine engineering
  - Graphene: Applications in medicine and electronics
  - Aerogels: Environmental applications



Introduction & Motivation

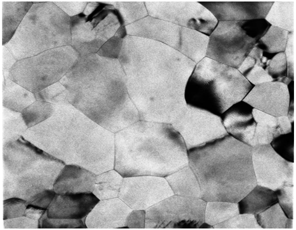
- Polycrystalline materials are composed of many crystalline parts that are randomly oriented with respect to each other.
- The material's properties are largely dependent on its microstructure.
- Material properties
  - conductivity
  - strength
  - hardness
  - corrosion resistance ...
- Material microstrucure properties
  - Grain size
  - Grain boundary distribution
  - Grain deformations
  - Chemical composition ...

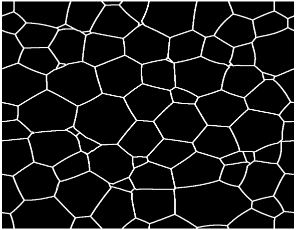
Introduction & Motivation

- Determining a materials properties through analysis of grain boundary structure is crucial to the development of new materials and subsequent advancement of engineering.
- Obtaining accurate measurements through imaging can be expensive and tedious.
  - Electron Backscatter Diffraction: Equipment and technician costs
  - Light optical microscopy: Requires preprocessing of the material and may affect measurement accuracy

Problem Statement

- Given an image of a polycrystalline material, can we implement an algorithm that will produce an accurate segmentation?
- i.e. Can we produce a binary image that accurately represents the grain boundary structure?

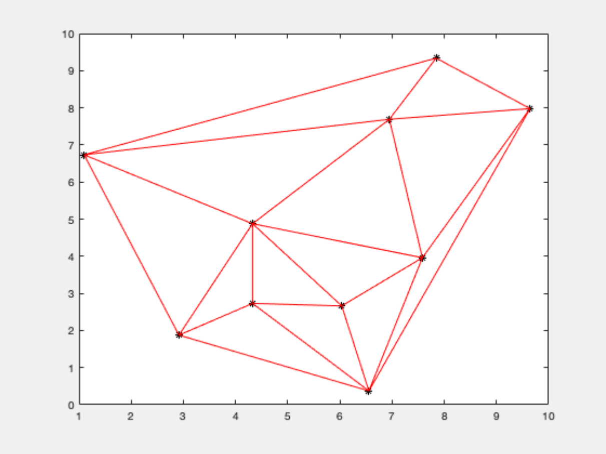




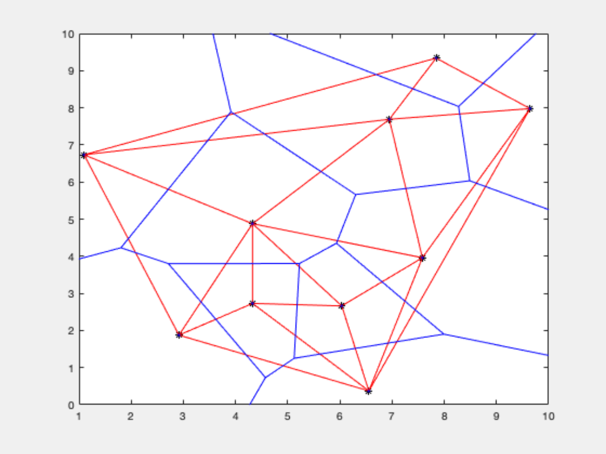
Voronoi Diagrams

- We can model grain boundary structure by using Voronoi Diagrams.
- Given a set of generating points $P = \{p_1, p_2, ..., p_n\}$, a plane is partitioned into n regions, $\{R_1, R_2, ..., R_n\}$, such that:
  - Each point p_i lies in exactly one region R_i.
  - For any point $q \notin P$ that lies in region R_i, the Euclidean distance from p_i to q will be shorter than the Euclidean distance from p_j to q $\forall j \neq i$

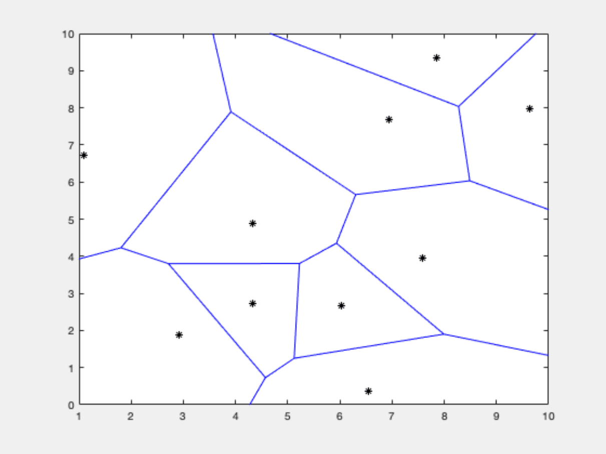
Voronoi Diagrams



Voronoi Diagrams



Voronoi Diagrams



Mean Curvature

- Mean-Curvature Flow of Voronoi Diagrams - Matt Elsey & Dejan Slepčev, 2014
- They were interested in gradient flow of Voronoi diagrams and proving universal bounds on coarsening rates.
- We followed a similar method while relaxing some constraints such as periodic boundary conditions. (more later)

Voronoi Diagram Energy

- Given a Voronoi Diagram with:
  - Generating points $P = \{p_1, p_2, ..., p_n\}$
  - Edges $S = \{s_1, s_2, ..., s_k\}$
  - Vertices $V = \{v_1, v_2, ..., v_m\}$.
- We define the energy of the Voronoi Diagram as:

$$E = \sum_{s_k \in S} Length(s_k) = \sum_{\substack{i,j \; s.t. \\ \overline{v_i v_j} = s_k \in S}} |v_i - v_j|$$

- Using this definition of energy, we can apply gradient descent on the generating points $P = \{p_1, p_2, ..., p_n\}$ and start to view some dynamics of the Voronoi diagrams.

Piece-wise Constant Mumford-Shah model

$$\sum_i \int_{R_i} (f(x,y) - c_i)^2 dxdy$$

- f(x,y): the target image's grayscale value at the pixel (x,y)
- c_i: the average pixel value in region R_i computed from f

Model

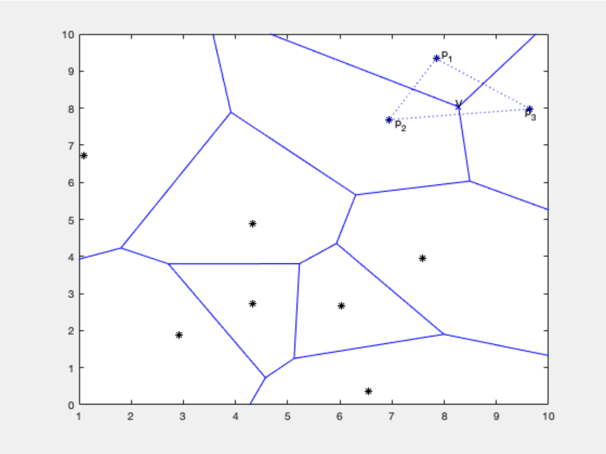
- Using the energy we defined earlier and the piece-wise constant Mumford-Shah we get:

$$E = \sum_{\substack{i,j\ s.t. \\ \overline{v_i v_j} = s_k \in S}} |v_i - v_j| + \sum_i \int_{R_i} (f(x,y) - c_i)^2 dxdy$$

Calculating Gradients: First Term

- Edges: $S = \{s_1, s_2, ..., s_k\}$
- Vertices: $V = \{v_1, v_2, ..., v_m\}$
- $\frac{\partial E}{\partial p_i} = \sum_{s_i \in S} \sum_{v_i \in vertex(s_i)} \frac{\partial Length(s_i)}{\partial v_i} \frac{\partial v_i}{\partial p_i}$
- $\frac{\partial length(s_i)}{\partial v_i} = [\frac{v_i(x) - v_j(x)}{Length(s_i)}, \frac{v_i(y) - v_j(y)}{Length(s_i)}]$
- How to calculate $\frac{\partial v_i}{\partial p_i}$?

Calculating Gradients: First Term



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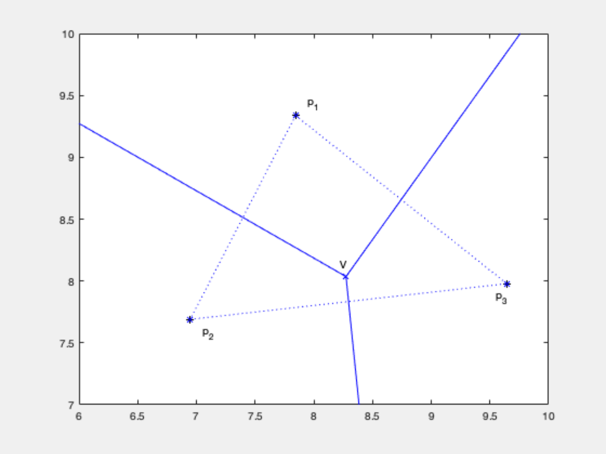
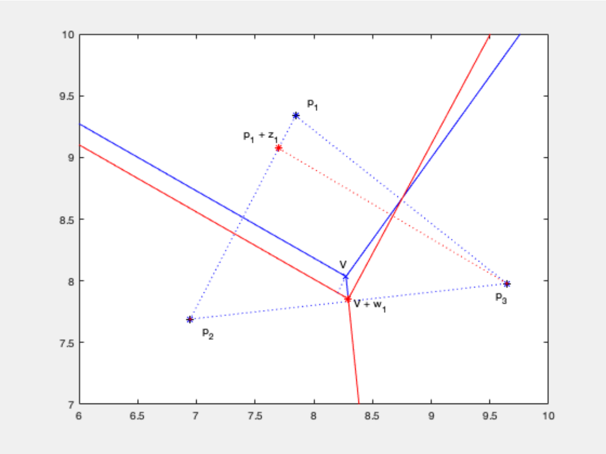


Calculating Gradients: First Term



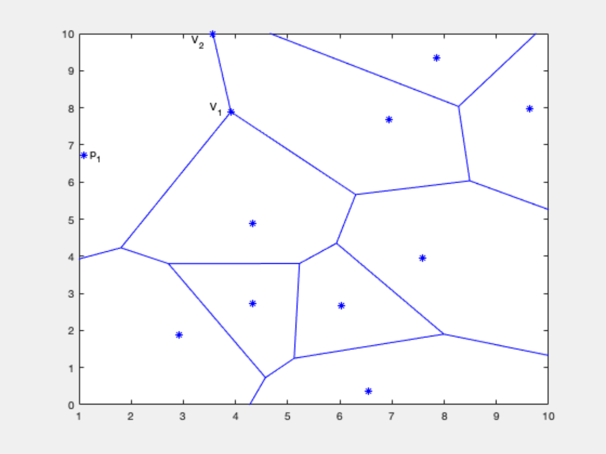
Calculating Gradients: First Term

- We now have a way to calculate changes in v_i for perturbations of p_i along two specific directions.
- Using a change of basis, we can get the gradient in terms of the standard basis
- $\frac{\partial v}{\partial p_1} = [w_1 \; w_2][s_1 \; s_2]^{-1}$

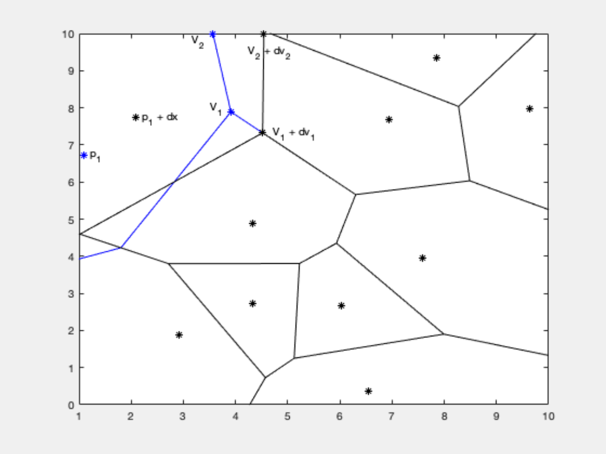
Calculating Gradients: Second Term

- $g(x, y) = (f(x, y) - c_i)^2$
- $\frac{\partial}{\partial p_i} \sum_i \int_{R_i} g(x, y) dxdy$
- note that after perturbing a center, the change in the integral comes from the part of the region that is changed.

Calculating Gradients: Second Term



Calculating Gradients: Second Term



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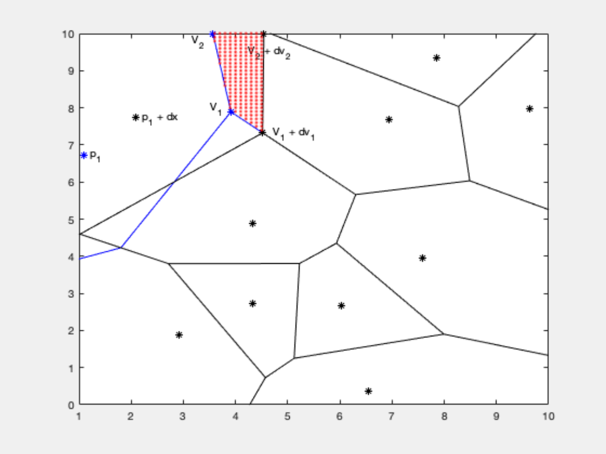


Calculating Gradients: Second Term

- $g(x, y) = (f(x, y) - c_i)^2$
- $\frac{\partial}{\partial p_i} \sum_i \int_{R_i} g(x, y) dxdy = \sum_i \int_{\partial R_i} g(s) v(s)^{\perp} ds$

$$\approx \sum_{R_i} \sum_{s_i \in edges(R_i)} \sum_{k=1}^{N} g(s) v_k(s)^{\perp} \Delta s$$

- $v(s)^{\perp} = \frac{L-r}{L} \frac{\partial v_i}{\partial p_1} \hat{n} + \frac{r}{L} \frac{\partial v_j}{\partial p_1} \hat{n}$

Gradient Descent Video

- play GD_Collision.mp4

Handling Topological Events

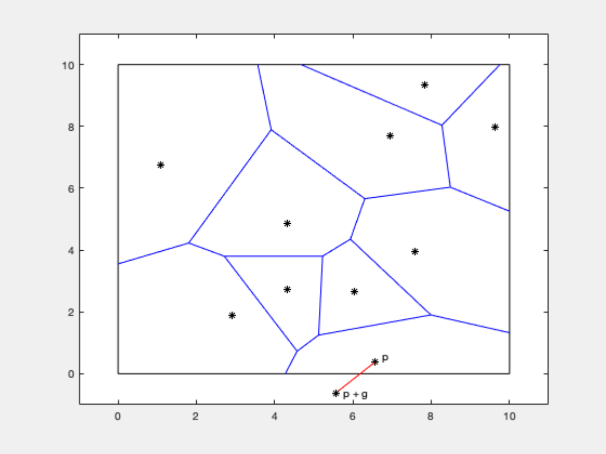
- Vertex collisions (handled well by the algorithm)
- Center collisions
- Vertices escape the boundary
- Center regions collapsed

Handling Topological Events - Center Collisions

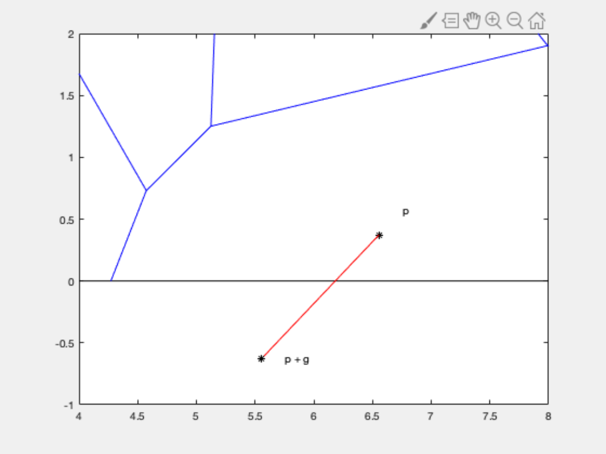
- Repulsion:

$$R(p_i, p_j) = R(d_{ij}) = e^{\frac{-1}{\theta^2 (r - d_{ij})^2}}$$

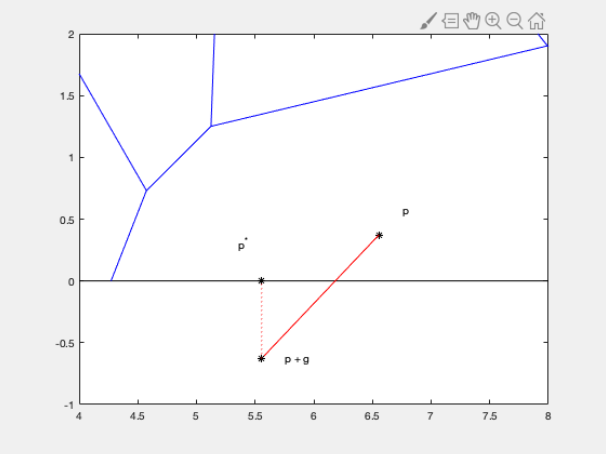
Handling Topological Events - Boundary Event



Handling Topological Events - Boundary Event



Handling Topological Events - Boundary Event



Handling Topological Events - Collapsed Regions

- $\text{Area}(R_i) \leq \tau \implies$ Removal of p_i

Gradient Descent Video

- play GD_RP.mp4

Image Segmentation Result

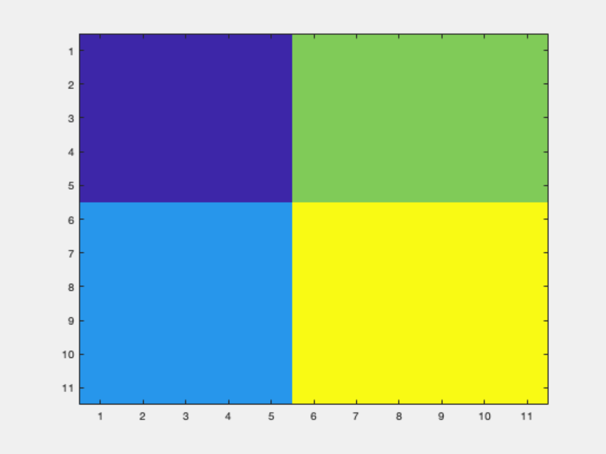


Image Segmentation Result

- play Sample_Segment.mp4

Conclusion and Future Work

- Our Model / Algorithm was able to properly handle a preliminary test case segmentation
- Future Challenges include
  - Non-Uniform Grain Colors
  - Non-Distinct Grain Colors
  - Generating Point Initialization - Location and Number of points

References

1. Elsey, Matt, and Dejan Slepčev. "Mean-curvature flow of Voronoi diagrams." Journal of Nonlinear Science 25.1 (2015): 59-85.
2. Trimby, P., et al. "Is fast mapping good mapping? A review of the benefits of high-speed orientation mapping using electron backscatter diffraction." Journal of microscopy 205.3 (2002): 259-269.
3. Tai, Xue-cheng, and Chang-hui Yao. "Image segmentation by piecewise constant Mumford-Shah model without Estimating the constants." Journal of Computational Mathematics, vol. 24, no. 3, 2006, pp. 435-443. JSTOR, www.jstor.org/stable/43693303.

Acknowledgments

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